<u>Recall</u>: Mean curv. of a regular surface $M \subseteq \mathbb{R}^3$

$$H(p) = \text{average of principle curvature.}$$

$$= \frac{1}{2} \frac{eG - 2fF + fE}{EG - F^2} \quad \text{where}$$

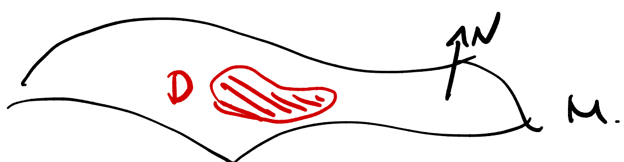
$$[\mathrm{II}]_X = \begin{bmatrix} e & f \\ f & g \end{bmatrix}, \quad [g]_X = \begin{bmatrix} E & F \\ F & G \end{bmatrix}.$$

second fundamental form. first fundamental form

Q: why care?? Any geometric meaning (instead linear algebra reason)??

<u>Objective</u>: Find geometric conseq. of M w/ $H \equiv 0$!!

Given $X : \mathcal{U} \to M \subseteq \mathbb{R}^3$, say $M = X(\mathcal{U})$. a regular parametrized surface. Let D be bdd open domain in $\mathcal{U}$.



N = normal vector of M.

Consider $X^t : \mathcal{U} \to M$ given by

$$X^t := X + t\,\underbrace{h \cdot N}_{\text{normal variation of } M.} \quad \text{where } h : \overline{D} \to \mathbb{R}. \text{ and } t \in (-\varepsilon, \varepsilon).$$

- <u>Claim</u>: X^t is a parametrization if $|t| \ll 1$.

pf: $\begin{cases} X^t_u = X_u + t h_u N + t h \cdot N_u \\ X^t_v = X_v + t h_v N + t h N_v \end{cases}$

$$\Rightarrow \quad X^t_u \times X^t_v = X_u \times X_v + O(t) \neq 0.$$

since $h \in C^\infty(\overline{D})$.

$\Rightarrow \{ X_u^t, X_v^t \}$ is linearly indep. if $|t| \ll 1$.

$\therefore$ $M_t = X^t(U)$ is a regular surface, $\forall |t| \ll 1$.

Consider $g^t = \langle , \rangle|_{T_pM_t}$.

$$g_{ij}^t = \langle X_i^t, X_j^t \rangle$$
$$= \langle X_i + t h_i N + t h N_i, X_j + t h_j N + t h N_j \rangle$$

$N \perp X_i$
$$= g_{ij} + t h (\langle N_i, X_j \rangle + \langle N_j, X_i \rangle) + O(t^2)$$
$$= g_{ij} - 2th \, \mathrm{II}_{ij} + O(t^2)$$

$\Leftrightarrow$ $[g^t] = [g] - 2th[\mathrm{II}] + O(t^2)$. (in terms of matrix)

i.e. $\partial_t g^t \big|_{t=0} = -2h\,\mathrm{II}$.

Lemma in linear algebra:

Jacobi formula: Given a variation of matrix $A(t)$, $t \in (-\varepsilon, \varepsilon)$

$$\frac{\partial}{\partial t} \det A = \mathrm{tr}\left(\mathrm{adj}(A) \cdot \frac{dA}{dt} \right). = \det A \cdot \mathrm{tr}\left(A^{-1} \frac{dA}{dt}\right)$$

if A^{-1} exists.

pf: (from Wiki)

case 1: $A(0) = I$ and $A'(0) = T \in GL(n, \mathbb{R})$.

$$\det(I + tT) = \det \begin{bmatrix} 1+tT_{11} & O(t) & \cdots \\ \vdots & 1+tT_{22} & \\ & \cdots & \ddots & 1+tT_{nn} \end{bmatrix}$$

$$= \prod_{j=1}^{n} (1 + tT_{jj}) + O(t)$$

$$= 1 + t \cdot \mathrm{tr}\, T + O(t).$$

$\therefore$ $\frac{d}{dt}\Big|_{t=0} \det A = \mathrm{tr}\, T$ in this case.

case 2 : $A(0) = A_0$ is invertible, $A'(0) = T \in GL(n, \mathbb{R})$

Define $\tilde{A}(t) = A_0^{-1} A(t)$ s.t. $\begin{cases} \tilde{A}(0) = I \\ \tilde{A}'(0) = A_0^{-1} T = \tilde{T} \end{cases}$

case 1 $\Rightarrow$ $\left.\frac{d}{dt}\right|_{t=0} \det \tilde{A} = \operatorname{tr} \tilde{T}$

$$\text{LHS} = \left.\frac{d}{dt}\right|_{t=0} \det (A_0^{-1} A(t)) = (\det A_0)^{-1} \cdot \left.\frac{d}{dt}\right|_{t=0} \det A(t).$$

$$\text{R.H.S} = \operatorname{tr} \tilde{T} = \operatorname{tr}(A_0^{-1} T)$$

$$\Rightarrow \frac{d}{dt} \det A = \det A \cdot \operatorname{tr}(A^{-1} \cdot \frac{d}{dt} A)$$

$$= \operatorname{tr}(\operatorname{adj} A \cdot \frac{d}{dt} A)$$

case 3 : Since invertible matrix are dense in $GL(n, \mathbb{R})$,

($\{ A \mid \det A \}$ is of measure zero in $GL(n, \mathbb{R})$)

case 2 $\Rightarrow$ case 3 by density. #

$\because$ $[g]_x$ is invertible,

$$\therefore \left.\frac{d}{dt}\right|_{t=0} \det [g^t] = \det [g] \cdot \operatorname{tr}([g]_x^{-1} \cdot [\mathrm{II}]_x) \cdot (-2h).$$

$$\Rightarrow \left.\frac{d}{dt}\right|_{t=0} dA^t = \left.\frac{d}{dt}\right|_{t=0} \left(\sqrt{\det g^t} \cdot du\, dv \right)$$

$$= \frac{1}{2}\sqrt{\det[g]} \cdot \mathrm{tr}\left([g]^{-1}[\mathrm{II}]\right) \cdot (-2h).$$

$$= -h \cdot \mathrm{tr}\big(\underbrace{[\mathrm{II}]\cdot[g]^{-1}}\big) \cdot dA.$$

Recall: $= \mathrm{tr}[S_p] =$ trace of shape operator

$$= -2h \cdot H \cdot \underbrace{dA}_{\text{area element}}.$$

$$\therefore \boxed{\left.\frac{d}{dt}\right|_{t=0} \mathrm{Area}(X^t(D)) = \int_D -2h\cdot H\sqrt{\det g}\, du\,dv}$$

|| first variation formula of Area.

Corollary:

$$\left.\frac{d}{dt}\right|_{t=0} \overbrace{\mathrm{Area}(X^t(D))}^{A(t)} = 0 \quad \forall \text{ normal variation on } D$$

iff $H \equiv 0$ on D.

Defn (terminology) A regular surface $M \subseteq \mathbb{R}^3$ is called minimal if $H \equiv 0$ on M.

Special case: If $M = \{(x, y, f(x,y))\}$,

what condition on $f \Rightarrow$ minimal ??

prop : M is minimal iff $\operatorname{div}\left(\frac{\nabla f}{(1+|\nabla f|^2)^{\frac{1}{2}}}\right) = 0$

pf: $X: \mathcal{U} \to M$ given by $X(u,v) = (u, v, f(u,v))$

$X_u = (1, 0, f_u)$ $\qquad$ $X_v = (0, 1, f_v)$

$X_{uu} = (0, 0, f_{uu})$ $\qquad$ $X_{vv} = (0, 0, f_{vv})$

$X_{uv} = X_{vu} = (0, 0, f_{uv})$.

$$\Rightarrow \quad [g]_X = \begin{bmatrix} 1+f_u^2 & f_u f_v \\ f_u f_v & 1+f_v^2 \end{bmatrix}$$

$$X_u \times X_v = \begin{bmatrix} i & j & k \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{bmatrix} = (-f_u, -f_v, 1)$$

$$\Rightarrow \quad N = \frac{(-f_u, -f_v, 1)}{\sqrt{1+f_u^2+f_v^2}}$$

$$\Rightarrow \quad \mathrm{II}_{ij} = \langle X_{ij}, N \rangle = \frac{f_{ij}}{\sqrt{1+f_1^2+f_2^2}}$$

$$\therefore \quad H = \frac{1}{2} \frac{1}{(1+f_u^2+f_v^2)^{3/2}} \Big((1+f_u^2) f_{vv} + (1+f_v^2) f_{uu} - 2 f_u f_v f_{uv} \Big)$$

#

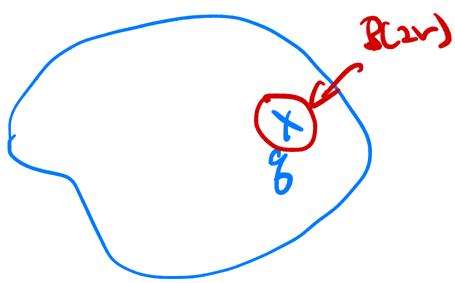
☆☆ In PDE, $\operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) = 0$ is an elliptic PDE ☆☆.

Variational viewpoint

prop: Let M be a regular surface.
If $X: \mathcal{U} \to M$ is a parametrization of M and $D \subset \mathcal{U}$ be a bdd open set in $\mathcal{U}$. Then $H=0$ on D iff $\left.\frac{d}{dt}\right|_{t=0} \text{Area}(X^t(D)) = 0$, $\forall$ normal variation of $X(D)$.

pf: ($\Rightarrow$): done by first variation formula.

($\Leftarrow$): If $\exists\, q \in D$ s.t. $H(q) \neq 0$.



WLOG, assume $H(q) > 0$

- $\exists\, r$ s.t. $H > 0$ on $B(2r)$
- taking φ smooth s.t.

$$\begin{cases} \varphi = 0 & \text{outside } B(2r) \\ \varphi = 1 & \text{on } B(r). \end{cases}$$

taking normal variation $X^t = X + t\varphi \cdot H$. cpt support

$$\Rightarrow \left.\frac{d}{dt}\right|_{t=0} \text{Area} = \int_D -2\varphi H^2 \sqrt{\det g}\, du\, dv$$

$$< 0 \quad \Rightarrow\Leftarrow \quad \#.$$

Rmk: cpt support: to be safe only s.t. the same arguement works for M.

Minimality under special coordinate :

In cplx coordinate, becomes $f \cdot |dz|^2$.

Fact (to be proved (OR not ??)) :

$\forall p \in M$, regular surface, $\exists X : \mathcal{U} \to M$ s.t. $[g]_X = \begin{bmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{bmatrix}$.

This coordinate is called the Isothermal coordinate.

prop : Under isothermal coordinate, if $N = \dfrac{X_u \times X_v}{\|X_u \times X_v\|}$,

then $X_{uu} + X_{vv} = 2\lambda^2 \underset{\text{mean curvature}}{H} \cdot N \in \mathbb{R}^3$.

pf: $\cdot$ $EG - F^2 = \lambda^2 \cdot \lambda^2 = \lambda^4$

$\langle X_{uu}, X_u \rangle = \frac{1}{2} \langle X_u, X_u \rangle_u = \lambda_u$

$\langle X_{vv}, X_u \rangle = - \langle X_v, X_{uv} \rangle = -\frac{1}{2} (\langle X_v, X_v \rangle)_u = -\lambda_u$

$\Rightarrow \langle X_{uu} + X_{vv}, X_u \rangle = 0$

interchang u, v $\Rightarrow \langle X_{uu} + X_{vv}, X_v \rangle = 0$

$\Rightarrow X_{uu} + X_{vv} \perp T_pM \Rightarrow X_{uu} + X_{vv} \parallel N$. since (2D)

$\Rightarrow X_{uu} + X_{vv} = \langle X_{uu} + X_{vv}, N \rangle N$

$= (e + g) N. = 2\lambda^2 H \cdot N$ #

where $H = \frac{1}{2} \dfrac{e+g}{\lambda^2}$

Why care ?? Because it means

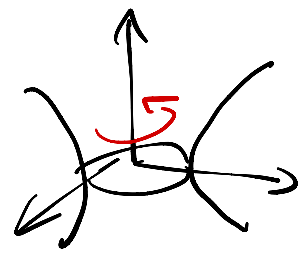
$H = 0 \Leftrightarrow \Delta X = 0$, i.e. coordinate fcn is harmonic !!

! cplx analysis enter !!

Example : $M =$ catenoid

$$X(u,v) = (a\cosh(v)\cos u,\ a\cosh(v)\sin u,\ av)$$

where $u \in (0, 2\pi)$, $v \in \mathbb{R}$.



$$X_{uu} = (-a\cosh(v)\cos u,\ -a\cosh(v)\sin u,\ 0)$$

$$X_{vv} = (a\cosh(v)\cos u,\ a\cosh(v)\sin u,\ 0)$$

$$\Rightarrow \quad X_{uu} + X_{vv} = 0 \quad \Rightarrow \text{minimal} \ \#$$

Example : (Helicoid)

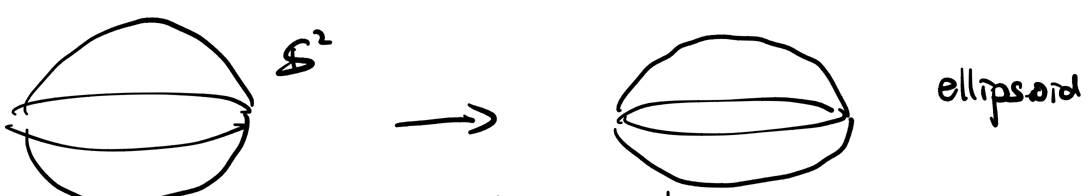
$$X(u,v) = (a\sinh(v)\cos u,\ a\sinh(v)\sin u,\ au)$$

$$X_{uu} + X_{vv} = 0$$

(Do my best please refer to wiki)

Rigidity of sphere in term of H ??

- $H \equiv$ const for S^2 but non-zero.
- Variation in term of fN is Too Random!!



want: strong volume inside.

Consider $X^t = X + tfN$

$$\Rightarrow \frac{d}{dt}\Big|_{t=0} \text{Area} = \int_M -2fH \cdot dA.$$

$H \equiv 0$

$\not\Uparrow$ from variation.

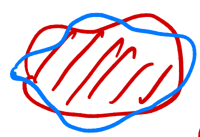
But now f is not arbitrary,

in particular f cannot be chosen to be $\varphi \cdot H$

Choice of f: $V(t)$ = Volume of solid inside of M.

co-area formula $\Rightarrow$ $\displaystyle \frac{d}{dt}\Big|_{t=0} V = \int_M f \cdot dA.$

$\Uparrow$



$\partial\Omega = M$ $\quad V(\Omega) = \int_\Omega dV$

$(u,v,w) \mapsto \underbrace{X(u,v) + w \cdot N}_{\text{parametrization of nbd of } \Omega}$

Hence, $\displaystyle \int_M fH\, dA = 0 \quad \forall f \text{ s.t. } \int_M f\, dA = 0.$

* if $H \equiv \text{const}$, then above holds (converse ??)

* Given $f \in C^\infty(M)$, take $\bar{f} = f - ⨍_M f\, dA$ s.t.

$$\int_M \bar{f}\, dA = 0 \Rightarrow \int_M \bar{f} H\, dA = 0, \quad \forall f \in C^\infty(M)$$

Now, taking $f = H$

$$\Rightarrow 0 = \int_M (H - \bar{H}) H\, dA = \int_M (H - \bar{H})^2\, dA$$

$\Rightarrow \quad H \doteq \overline{H}$ on M

i.e. $\quad H \equiv \text{const}$ on M #

$\Downarrow$

<u>prop</u>: Suppose M = critical pt of Area functional which preserve volume inside, then $H \equiv \text{const}$ in M.

☆ we call M a CMC surface.

☆ the discussion above also works in higher dimension.